A Level Mathematics A

H240/01 Pure Mathematics

Question Set 5

1 **(a)** For a small angle θ, where θ is in radians, show that $2\cos\theta + (1-\tan\theta)^2 \approx 3 - 2\theta$. **[3]**

1\. a) $2\cos\theta + (1-\tan\theta)^2$

$\approx 2(1-\frac{\theta^2}{2}) + (1-\theta)^2$

$\approx 2 - \cancel{\theta^2} + 1 - 2\theta + \cancel{\theta^2}$

$\approx 3 - 2\theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta$

(b) Hence determine an approximate solution to $2\cos\theta + (1-\tan\theta)^2 = 28\sin\theta$. **[2]**

b) $2\cos\theta + (1-\tan\theta)^2 = 28\sin\theta$

$3 - 2\theta \approx 28\sin\theta$

$3 - 2\theta = 28\theta$

$\sin\theta \approx \theta$

$30\theta = 3$

$\theta = \frac{1}{10}$ rad

2 A cylindrical metal tin of radius r cm is closed at both ends. It has a volume of $16000\pi\,\text{cm}^3$.

(a) Show that its total surface area, $A\,\text{cm}^2$, is given by $A = 2\pi r^2 + 32000\pi r^{-1}$. **[4]**

2\. $V = 16000\pi$ $\quad r$

a) $V = \pi r^2 h = 16000\pi$

$h = \frac{16000\cancel{\pi}}{\cancel{\pi}r^2} = \frac{16000}{r^2}$

$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{16000}{r^2}\right)$

$= 2\pi r^2 + 2\pi \times 16000 r^{-1}$

$A = 2\pi r^2 + 32000\pi r^{-1}$

(b) Use calculus to determine the minimum total surface area of the tin. You should justify that it is a minimum. **[6]**

b) $\frac{dA}{dr} = 4\pi r - 32000\pi r^{-2}$

for minimum, $\frac{dA}{dr} = 0$

$4\pi r - 32000\pi r^{-2} = 0$

$\frac{32000\cancel{\pi}}{r^2} = 4\cancel{\pi} r$

$32000 = 4r^3$

$r^3 = 8000$

$r = 40\sqrt{5}$

$A = 2\pi(40\sqrt{5})^2 + \frac{32000\pi}{40\sqrt{5}} = 51389.5\,cm^2 \approx 51000\,cm^3$

$\frac{d^2A}{dr^2} = 4\pi + 64000\pi r^{-3}$

$r = 40\sqrt{5}$

$4\pi + \frac{64000\pi}{(40\sqrt{5})^3} > 0$ positive so minimum

3 Prove by contradiction that there is no greatest multiple of 5. **[3]**

3 assume that N is greatest multiple of 5

if $N = 5a$, $N+5$ is $5a+5 = 5(a+1)$ which is divisible by 5 thus multiple of 5 which is greater $[\,5(a+1) > 5a\,]$

hence, contradicts original statement that N is greatest multiple of 5

4 Two students, Anna and Ben, are starting a revision programme. They will both revise for 30 minutes on Day 1. Anna will increase her revision time by 15 minutes for every subsequent day. Ben will increase his revision time by 10% for every subsequent day.

(a) Verify that on Day 10 Anna does 94 minutes more revision than Ben, correct to the nearest minute. **[3]**

4. Anna

30 $\xrightarrow{+15}$ $\xrightarrow{+15}$

Ben

30 $\xrightarrow{\times\frac{110}{100}}$ $\xrightarrow{\times\frac{110}{100}}$

a) Anna arithmetic sequence

$u_n = a + (n-1)d$

$u_{10} = 30 + (10-1) \times 15 = 165 \text{ minutes}$

Ben geometric sequence

$u_n = ar^{n-1}$

$u_{10} = 30 \times \left(\frac{110}{100}\right)^{10-1} = 70.7 \text{ minutes}$

$165 - 70.7 = 94.3 \approx$ 94 minutes

Let Day X be the first day on which Ben does more revision than Anna.

(b) Show that X satisfies the inequality $X > \log_{1.1}(0.5X + 0.5) + 1$. **[3]**

day X when Ben > Anna

b) $u_x = 30 \times \left(\frac{110}{100}\right)^{x-1} > u_x = 30 + (x-1) \times 15$

$30\left(\frac{110}{100}\right)^{x-1} > 30 + 15x - 15$

$30\left(\frac{110}{100}\right)^{x-1} > 15 + 15X$

$\left(\frac{110}{100}\right)^{x-1} > \frac{1}{2} + \frac{1}{2}X$

$(x-1)\log_{1.1}\left(\frac{110}{100}\right) > \log_{1.1}(0.5 + 0.5x)$

$X - 1 > \log_{1.1}(0.5 + 0.5X)$

$X > \log_{1.1}(0.5X + 0.5) + 1$

(c) Use the iterative formula $x_{n+1} = \log_{1.1}(0.5x_n + 0.5) + 1$ with $x_1 = 10$ to find the value of X.

You should show the result of each iteration. [3]

c) $x_{n+1} = \log_{1.1}(0.5x_n + 0.5) + 1$

$x_1 = 10$

$x_2 = \log_{1.1}(0.5 \times 10 + 0.5) + 1 = 18.89$

$x_3 = \log_{1.1}(0.5 \times 18.89 + 0.5) + 1 = 25.10$

$x_4 = \log_{1.1}(0.5 \times 25.10 + 0.5) + 1 = 27.95$

$x_5 = \log_{1.1}(0.5 \times 27.95 + 0.5) + 1 = 29.04$

$x_6 = \log_{1.1}(0.5 \times 29.04 + 0.5) + 1 = 29.43$

$x_7 = \log_{1.1}(0.5 \times 29.43 + 0.5) + 1 = 29.56$

$x_8 = \log_{1.1}(0.5 \times 29.56 + 0.5) + 1 = 29.61$

$x_9 = \log_{1.1}(0.5 \times 29.61 + 0.5) + 1 = 29.62$

$x_{10} = \log_{1.1}(0.5 \times 29.62 + 0.5) + 1 = 29.63$

$x_{11} = \log_{1.1}(0.5 \times 29.63 + 0.5) + 1 = 29.63$

$\vdots$

≈ 30

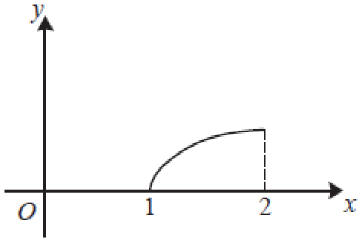
(d) (i) Give a reason why Anna's revision programme may not be realistic. [1]

d) (i) - eventually, the time predicted will exceed 1440 minutes which is over a full day that cannot happen

(ii) Give a **different** reason why Ben's revision programme may not be realistic. [1]

(ii) - Ben cannot revise for a long time without a break and would invest time for other daily activities so not realistic

5



The diagram shows the curve $y = \sin\left(\frac{1}{2}\sqrt{x-1}\right)$, for $1 \leqslant x \leqslant 2$.

(a) Use rectangles of width 0.25 to find upper and lower bounds for $\int_1^2 \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx$. Give your answers correct to 3 significant figures. [4]

5. $y = \sin\left(\frac{1}{2}\sqrt{x-1}\right)$ $\quad 1 \leq x \leq 2$

a) $x: 1 \sim 1.25 \Rightarrow 0.25$

$y = \sin\left(\frac{1}{2}\sqrt{1-1}\right) = 0$

$y = \sin\left(\frac{1}{2}\sqrt{1.25-1}\right) = 4.36 \times 10^{-3}$

lower bound: $(0 + 4.36 \times 10^{-3}) \times 0.25 \times \frac{1}{2} = 5.45 \times 10^{-4}$

lower bound $= 5.45 \times 10^{-4}$

$x: 1.75 \sim 2 \Rightarrow 0.25$

$y = \sin\left(\frac{1}{2}\sqrt{1.75-1}\right) = 7.56 \times 10^{-3}$

$y = \sin\left(\frac{1}{2}\sqrt{2-1}\right) = 8.73 \times 10^{-3}$

upper bound: $(7.56 \times 10^{-3} + 8.73 \times 10^{-3}) \times 0.25 \times \frac{1}{2}$

$= 2.04 \times 10^{-3}$

upper bound $= 2.04 \times 10^{-3}$

(b) **(i)** Use the substitution $t = \sqrt{x-1}$ to show that $\int \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx = \int 2t\sin\left(\frac{1}{2}t\right)dt$. [3]

b)(i) $\int \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx$

$t = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$

$\frac{dt}{dx} = \left(\frac{1}{2}(x-1)^{-\frac{1}{2}}\right)(1) = \frac{1}{2\sqrt{x-1}}$

$dx = 2\sqrt{x-1}\, dt$

$= \int \sin\left(\frac{1}{2}\sqrt{x-1}\right) \times 2\sqrt{x-1}\, dt$

$= \int \sin\left(\frac{1}{2}t\right) \times 2t\, dt$

$= \int 2t\sin\left(\frac{1}{2}t\right)dt$

(ii) Hence show that $\int_1^2 \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx = 8\sin\frac{1}{2} - 4\cos\frac{1}{2}$. [4]

(ii) $\int_1^2 \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx$

$t = \sqrt{x-1}$

$t = \sqrt{2-1} = 1$

$t = \sqrt{1-1} = 0$

$= \int_0^1 2t\sin\left(\frac{1}{2}t\right)dt$

$u = 2t \qquad u' = 2$

$v = -2\cos\left(\frac{1}{2}t\right) \qquad v' = \sin\left(\frac{1}{2}t\right)$

$= \left[(2t)\left(-2\cos\left(\frac{1}{2}t\right)\right) - \int\left(-2\cos\left(\frac{1}{2}t\right)\right)(2)\,dt\right]_0^1$

$= \left[-4t\cos\left(\frac{1}{2}t\right) - \int -4\cos\left(\frac{1}{2}t\right)dt\right]_0^1$

$= \left[-4t\cos\left(\frac{1}{2}t\right) + 4\int\cos\left(\frac{1}{2}t\right)dt\right]_0^1$

$= \left[-4t\cos\left(\frac{1}{2}t\right) + 4\left(2\sin\left(\frac{1}{2}t\right)\right)\right]_0^1$

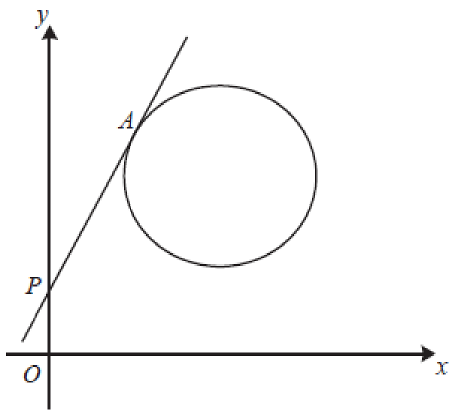
$= \left[-4t\cos\left(\frac{1}{2}t\right) + 8\sin\left(\frac{1}{2}t\right)\right]_0^1$

$= \left[-4(1)\cos\left(\frac{1}{2}\times 1\right) + 8\sin\left(\frac{1}{2}\times 1\right)\right] - \left[-4(0)\cos\left(\frac{1}{2}\times 0\right) + 8\sin\left(\frac{1}{2}\times 0\right)\right]$

$= -4\cos\frac{1}{2} + 8\sin\frac{1}{2} + 0 - 0$

$= 8\sin\frac{1}{2} - 4\cos\frac{1}{2}$

6



The diagram shows a circle with equation $x^2+y^2-10x-14y+64=0$. A tangent is drawn from the point $P(0, 2)$ to meet the circle at the point A. The equation of this tangent is of the form $y=mx+2$, where m is a constant **greater than** 1.

(a) **(i)** Show that the x-coordinate of A satisfies the equation $(m^2+1)x^2-10(m+1)x+40=0$. **[2]**

6. $x^2+y^2-10x-14y+64=0$

$y=mx+2$ $\quad m>1 \quad P(0,2)$

a) (i) $x^2+(mx+2)^2-10x-14(mx+2)+64=0$

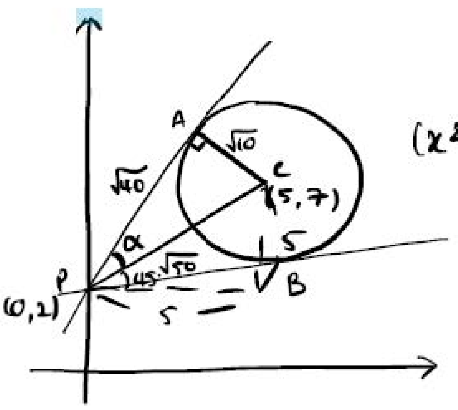
$x^2+m^2x^2+4mx+4-10x-14mx-28+64=0$

$(m^2+1)x^2+(-10m-10)x+40=0$

$(m^2+1)x^2-10(m+1)x+40=0$

(ii) Hence determine the equation of the tangent to the circle at A which passes through P. **[4]**

(ii)



$x^2+y^2-10x-14y+64=0$

$(x^2-10x)+(y^2-14y)+64=0$

$(x^2-10x+25-25)+(y^2-14y+49-49)+64=0$

$(x-5)^2+(y-7)^2-25-49+64=0$

$(x-5)^2+(y-7)^2-10=0$

$(x-5)^2+(y-7)^2=10=(\sqrt{10})^2$

$C(5,7)$

length of PC $=\sqrt{(5-0)^2+(7-2)^2}=\sqrt{25+25}=\sqrt{50}$

$(\sqrt{50})^2=(\sqrt{10})^2+AP^2 \qquad AP=\sqrt{40}$

$m=\tan(45^\circ+\alpha^\circ) \qquad \tan\alpha=\dfrac{\sqrt{10}}{\sqrt{40}}=\dfrac{1}{2}$

$=\dfrac{\tan 45^\circ+\tan\alpha^\circ}{1-\tan 45^\circ \tan\alpha^\circ}$

$=\dfrac{1+\frac{1}{2}}{1-1\times\frac{1}{2}}=\dfrac{\frac{3}{2}}{\frac{1}{2}}=3 \qquad y=3x+2$

A second tangent is drawn from P to meet the circle at a second point B. The equation of this tangent is of the form $y = nx + 2$, where n is a constant **less than** 1.

(b) Determine the exact value of $\tan APB$. **[4]**

b) $n = \tan(45° - \alpha°)$

$$= \frac{\tan 45° - \tan\alpha°}{1 + \tan 45° \tan\alpha°}$$

$$= \frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

$$\tan APB = \tan 2\alpha = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha \tan\alpha} = \frac{\frac{1}{2} + \frac{1}{2}}{1 - \frac{1}{2} \times \frac{1}{2}} = \frac{4}{3}$$

Total Marks for Question Set 5: 50 Marks